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Numerical Solution of the Collisionless Boltzmann Equation for Stellar Systems*

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This paper presents a numerical procedure for the direct numerical solution of the collisionless Boltzmann equation appropriate to stellar dynamics. The velocity distribution need not be isotropic but spherical spatial symmetry is assumed. The difference scheme uses a second-order implicit scheme combined with an analog of the S_n method of Carlson. Inclusion of collisions, external force fields, and the two-body distribution function will be straightforward. The results for the collisionless system are accurate up to 10 crossing times. Both known analytic solutions and nonequilibrium models have been tested. An appendix gives the absicissas and weight factors for the Gaussian integration of $\int_0^1 f(x)x^2\,dx$.

I. INTRODUCTION

The dynamical evolution of a system of N particles that move under the influence of their own gravitational field is an old one in the area of stellar dynamics. The most frequently used methods of solution are (i) direct numerical integration of the equations of motion for each particle, (ii) a Monte Carlo calculation for the orbits of the particles, or (iii) a fluid dynamical approach, wherein moments of the Boltzmann equation define the density, pressure tensor, etc. Numerical integration of the Newtonian equations of motion, while intrinsically the simplest and most direct course, is limited to small N (e.g., $N \leqslant 500$). The Monte Carlo approach is very expensive and accurate for only a few relaxation times. The fluid dynamical method is usually limited to systems with a nearly Maxwellian velocity distribution. Hence, there is a need for a technique that can compete with the Monte Carlo calculations (in terms of speed and accuracy) and not be limited by either the assumption that N is small, or that the velocity distribution is nearly Maxwellian.

The method presented in this paper allows one to integrate directly the collisionless Boltzmann equation. It has been formulated in such a way that the inclusion of collisions, of external force fields, and of the two-body distribution

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function (which yields information about particle–particle correlations) can be simply effected. Thus, the evolution of globular clusters, tidally limited elliptical galaxies, and clusters of galaxies may be studied in detail. Most of these objects are nearly spherically symmetric, but their individual constituents are not necessarily without angular momentum. Hence, we discuss only the (spatially) spherically symmetric case with an anisotropic velocity distribution. No other assumptions are made regarding the system structure.

In the following section, the proposed difference scheme is presented. Section III contains the results of numerical tests using time-independent (analytic) solutions. In Section IV, the technique is further illustrated by studying a nonequilibrium situation. Comparisons are made to other, similar work. The Appendix contains the numbers for the Gaussian approximation to

$$\int_0^1 f(x)\, x^2\, dx = \sum_{j=1}^{16} b_j f(x_j). \tag{1}$$

This extends the work of Fishman [3].

II. The Difference Scheme

A. *Analytic Preliminaries*

The collisionless Boltzmann equation has the form

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial r} + \frac{dU}{dt}\frac{\partial f}{\partial U} + \frac{dV}{dt}\frac{\partial f}{\partial V} + \frac{dW}{dt}\frac{\partial f}{\partial W} = 0, \tag{2}$$

where f is the single particle distribution function, (U, V, W) is the velocity vector, and spatial spherical symmetry has been assumed. The components of the vector (U, V, W) and $(dU/dt, dV/dt, dW/dt)$ are

$$U = dr/dt, \qquad dU/dt = -a(r, t) + (V^2 + W^2)/r, \tag{3}$$

$$V = r\, d\theta/dt, \qquad dV/dt = -UV/r + W^2 \cot\theta/r, \tag{4}$$

and

$$W = r \sin\theta\, d\phi/dt, \qquad dW/dt = -UW/r - VW \cot\theta/r \tag{5}$$

[6], where $a(r, t)$ is the negative gradient of the self-consistent gravitational potential, ϕ is the spatial azimuthal coordinate, and θ is the spatial latitudinal coordinate. If spherical velocity coordinates (v, Θ, Φ) are introduced, viz

$$U = v \cos\Theta \equiv \mu v, \tag{6}$$

$$V = v \sin\Theta \cos\Phi, \tag{7}$$

$$W = v \sin\Theta \sin\Phi, \tag{8}$$

and the restriction that the velocity distribution is everywhere axisymmetric with respect to the radius vector is imposed, then (2) takes the form in (9).

$$\frac{\partial f}{\partial t} + \mu v \frac{\partial f}{\partial r} - a \left[\mu \frac{\partial f}{\partial v} + \frac{(1-\mu^2)}{v} \frac{\partial f}{\partial \mu}\right] + \frac{(1-\mu^2)\, v}{r} \frac{\partial f}{\partial \mu} = 0. \tag{9}$$

If G is the gravitational constant and m the particle mass, then

$$a(r, t) \equiv GM(r, t)/r^2 = (Gm/r^2) \int_0^r 4\pi y^2\, dy \int_0^\infty 2\pi v^2\, dv \int_{-1}^{+1} f(y, v, \mu, t)\, d\mu. \tag{10}$$

As a last preliminary step, the physical variables are scaled and the dimensionless variables, x, u, τ, g, and α are introduced. If N is the total number of particles, u_m the dimensionless maximum speed, R the total radius, and β related to the kinetic temperature ($=1/\mathrm{kT}$) when the velocity distribution is Maxwell–Boltzmann) then

$$x = r/R, \qquad t = R(m\beta)^{1/2}\, \tau/u_m\,, \qquad u = (m\beta)^{1/2}\, v/u_m\,, \tag{11}$$

and

$$\alpha = aRm\beta/u_m^2, \qquad f = N(m\beta)^{3/2}\, g/(8\pi^2 R^3). \tag{12}$$

When $g(x, u, \mu, 0)$ is specified, its further evolution is given by (13).

$$\frac{\partial g}{\partial \tau} + \mu u \frac{\partial g}{\partial x} - \alpha \left[\mu \frac{\partial g}{\partial u} + \frac{(1-\mu^2)}{u} \frac{\partial g}{\partial \mu}\right] + \frac{(1-\mu^2)\, u}{x} \frac{\partial g}{\partial \mu} = 0. \tag{13}$$

B. *The Difference Scheme*

There appear to be two straightforward ways to handle the u, μ variation. One is to write

$$g(x, u, \mu, \tau) = \sum_{l,n=0}^{\infty} G_{ln}(x, \tau)\, P_l(\mu)\, L_n(u), \tag{14}$$

where P_l is the lth Legendre polynomial and L_n is the nth Laguerre polynomial. This proved unsatisfactory. Instead, we consider a variation of the S_n method of Carlson [1] (see [7]). Let $\{\mu_j\}$ ($\mu_j \in (-1, 1)$, $j = 1,\ldots, J$), $\{u_k \in (0, 1)$, $k = 1,\ldots, K)$ be a division of the μ, u ranges. Assume one only knows g at these net points. If we abbreviate $g(x, u = u_k\,, \mu = \mu_j\,, \tau)$ by $g_{jk}(x, \tau)$ then (13) is replaced by the JK differential equations

$$\frac{\partial g_{jk}}{\partial \tau} + \mu_j u_k \frac{\partial g_{jk}}{\partial x} = \alpha \left[\mu_j \left(\frac{\partial g_j}{\partial u}\right)_k + \frac{(1-\mu_j^2)}{u_k} \left(\frac{\partial g_k}{\partial \mu}\right)_j\right] - \frac{(1-\mu_j^2)\, u_k}{x} \left(\frac{\partial g_k}{\partial \mu}\right)_j$$

$$= h_{jk}(x, \tau); \qquad j = 1, 2,\ldots, J;\; k = 1, 2,\ldots, K. \tag{15}$$

The derivatives with respect to μ and u are evalutated using a three-point Lagrange interpolation formula. To solve these JK equations, a three-level implicit difference scheme is employed. The spatial grid is

$$x(1) = 0, \qquad x(2) = \Delta x/2, \qquad x(i) = (i-2)\,\Delta x, \qquad \text{for } i = 3, 4, \ldots, I. \tag{16}$$

Denote $g_{jk}(x = x(i), \tau = n\,\Delta\tau)$ by $g_{jk}^n(i)$. Then, the most common term (there are slight variations for $i = 1, 2, 3, I$; see (20)) of the difference scheme is

$$\begin{aligned}[3g_{jk}^{n+1}(i) - 4g_{jk}^n(i) + g_{jk}^{n-1}(i)]/(2\,\Delta\tau)\\ + (\mu_j u_k/2\,\Delta x)[g_{jk}^{n+1}(i+1) - g_{jk}^{n+1}(i-1)] = h_{jk}^n(i).\end{aligned} \tag{17}$$

To compute $g_{jk}^{n+1}(1)$ (since $h_{jk}^n(1)$ has the form of 0/0) the fact that the particle density near $x = 0$ is a continuous function of x is used. In particular, the three-point Lagrange interpolation formula centered at $x = x(3)$ is extrapolated to $x = 0$. Thus,

$$g_{jk}^{n+1}(1) = (8/3)\,g_{jk}^{n+1}(2) - 2g_{jk}^{n+1}(3) + (1/3)\,g_{jk}^{n+1}(4). \tag{18}$$

Let $b_{jk} = \mu_j u_k\,\Delta\tau/(3\,\Delta x)$ and drop obvious jk subscript pairs. Then, in matrix notation, the full difference scheme can be written

$$Ag^{n+1} = (2\,\Delta\tau/3)\,h^n + [4g^n - g^{n-1}]/3, \qquad I \geqslant i > 1, \tag{19}$$

where

$$A = \begin{bmatrix} -2b & 1 & 2b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -8b/3 & 1+2b & 2b/3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -b & 1 & b & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & 1 & b & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -b & 1 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & b & -4b & 1+3b \end{bmatrix} \tag{20}$$

This matrix is augmented by (18) to form A' (the augmentation is in the first row of A').

Attempts to solve this system by using recurrence methods or relaxation techniques have proved unsatisfactory due to instability. Moreover, it is clear that the absolute value of $\det(A') \simeq 1$, and that the inverse of A' has significant elements only along its main diagonal and the three diagonals above and below the main one. Furthermore, since A' is independent of both i and n once the μ, u set and $\Delta\tau/\Delta x$ have been chosen, the JK inversions need be performed only once each.

Also, one need store just $(7I - 12)\,JK$ elements instead of I^2JK elements. Therefore, a more satisfactory method consists of computing and storing the inverse of A' once and for all, for every value of b_{jk}.

The choice of $\{\mu_j\}$ and $\{u_k\}$ ideally should maximize the accuracy of the μ, u derivatives and the calculation of α, (21), by Gaussian quadrature.

$$\alpha(x, \tau) = (GNm^2\beta\mu_m/R) \int_0^x (y/x)^2\, dy \int_0^1 u^2\, du \int_{-1}^{+1} g\, d\mu. \tag{21}$$

In practice, the μ, u set is the one for the Gaussian quadrature of the μ, u integrals in (21) (see the Appendix).

III. Numerical Tests

There exists an infinity of analytical solutions to the time-independent collisionless Boltzmann equation. These can be used to test the accuracy of the difference scheme. Of special astrophysical interest is the isothermal sphere, polytropes [2] and generalized polytropes [4]. There are always three conserved integrals: They are the total mass M, the sum of the squares of the angular momentum of the particles L^2, and the total energy E. If $\{w_j\}$, $\{w_k'\}$ are the weights in the Gaussian quadrature formulas for the μ, u integrations, then,

$$M = mN = mNu_m^3 \sum_{j,k} w_j w_k' \int_0^1 x^2 g_{jk}\, dx, \tag{22}$$

$$L^2 = (NmR^2u_m^5/\beta) \sum_{j,k} w_j w_k'(1 - \mu_j^2)\, u_k^2 \int_0^1 x^4 g_{jk}\, dx, \tag{23}$$

and

$$E = K + \Omega, \tag{24}$$

where

$$K = (Nu_m^5/2\beta) \sum_{j,k} w_j w_k' u_k^2 \int_0^1 x^2 g_{jk}\, dx, \tag{25}$$

$$\Omega = -(GM^2u_m^6/R) \sum_{j,k} w_j w_k' \int_0^1 x g_{jk}(x, \tau) \sum_{r,s} w_r w_s' \int_0^x y^2 g_{rs}(y, \tau)\, dy\, dx. \tag{26}$$

Also of interest in describing the cluster structure is the moment of inertia I and the velocity moments $\langle \mu u \rangle$ ($\equiv 0$), $\langle \mu^2 u^2 \rangle$, and $\langle (1 - \mu^2)\, u^2/2 \rangle$. The average of μu must vanish because there is no net motion of the center of mass. The ratio $i \equiv \langle (1 - \mu^2)\, u^2/2 \rangle / \langle \mu^2 u^2 \rangle$ determines the anisotropy of the velocity distribution ($i = 1$ for an isotropic distribution). Finally, for systems satisfying the virial theorem, $K/\Omega = -\frac{1}{2}$.

A. *The Isothermal Sphere*

Here, as in all numerical tests described in this section, $J = K = 16$, $I = 32$ ($\Delta x = 1/30$), and $\Delta\tau = 0.01$. If $\epsilon = (U^2 + V^2 + W^2)/2 + \phi(r)$ is the particle energy per unit mass then the distribution is

$$f = B\exp(-m\beta\epsilon), \tag{27}$$

where ϕ is the gravitational potential and B is a constant. Thus,

$$g(x, u, \mu, 0) = B'z(x)\exp[-u^2u_m{}^2/2], \tag{28}$$

where $z(x)$ satisfies

$$(d/dx)[x^2(d\ln z/dx)] = -x^2z; \qquad z(0) = 1,\ z'(0) = 0. \tag{29}$$

In this case, u_m is really infinite. However, because of the rapid decrease of the exponential $u_m = 5$ or 10 covers the speed range of interest. The results in Table I used $u_m = 10$ (the crossing time $T = R/(\langle u^2\rangle)^{1/2} = 5.35 \times 10^{-2}$ so $\Delta\tau = 0.187T$) and extend to $x = 6$ from (29). Isothermal spheres have a well-known instability at $x = 6.45$.

The units for Table I are $10^{12}M_0$ for mass, $10Mpc$ for distance, and 10^{10} yr for time. These are appropriate for the study of clusters of galaxies and $G = 4.495 \times 10^{-4}$ in these units.

The difference scheme is accurate for a time $\sim 10T$ when a flow of mass toward the center renders the results useless.

B. *Polytropes*

The distribution function is [2]

$$\begin{aligned} f &= B(E_1 - \epsilon)^{n-3/2}, & \epsilon &\leqslant E_1, \\ &= 0, & \epsilon &\geqslant E_1, \end{aligned} \tag{30}$$

where ϵ is as before and B, E_1 are constants. It is known that polytropes with $n \geqslant \frac{3}{2}$ are stable. In this case

$$\begin{aligned} g(x, u, \mu, 0) &= B'[w(x) - u^2]^{n-3/2}, & w(x) &\geqslant u^2 \\ &= 0, & w(x) &\leqslant u^2, \end{aligned} \tag{31}$$

where w satisfies

$$(d/dx)(x^2(dw/dx)) = -x^2w^n; \qquad w(0) = 1,\ w'(0) = 0. \tag{32}$$

The numerical accuracy is illustrated by using the $n = 5$ polytrope (also called Plummer's model) since (32) can be analytically solved in this case ($w(x) = 1/(1 + x^2/3)^{1/2}$).

TABLE I

Numerical Tests of Analytic Solutions[a]

Model	Time	Crossing time	Mass	Energy	Angular momentum	Virial ratio	Moment of inertia	Radial velocity of the c.m.	Anisotropy ratio	Central density
	τ	T	M	E	L^2	K/Ω	I	$\langle \mu u \rangle$	i	ρ_c
Isothermal	0	5.35 − 2	1.00 + 4	−1.62 + 4	9.39 + 3	−5.00 − 1	4.35 + 3	−3.04 − 9	1.00	1.20 + 4
Sphere	5.5 − 1		1.01 + 4	−1.60 + 4	9.57 + 3	−5.13 − 1	4.34 + 3	−1.88 − 1	1.12	1.09 + 5
$n = 5$	0	2.47 − 2	1.00 + 4	−1.92 + 4	5.11 + 3	−5.00 − 1	3.42 + 3	−3.66 − 9	1.00	5.39 + 4
Polytrope	3.5 − 1		1.01 + 4	−1.93 + 4	5.29 + 3	−5.08 − 1	3.46 + 3	−2.13 − 2	0.95	5.38 + 4
Generalized	0	3.33 − 2	1.00 + 4	−1.27 + 4	1.32 + 4	−5.00 − 1	6.90 + 3	−1.93 − 9	1.50	0
Polytrope	5.5 − 1		1.04 + 4	−1.37 + 4	1.39 + 4	−4.97 − 1	7.21 + 3	−2.00 − 1	1.39	7.43 + 3

[a] The notation 1.23 − 4 means 1.23×10^{-4}.

The results are in Table I ($T = 2.47 \times 10^{-2}$ so $\Delta\tau = 0.405T$; note that T is always independent of u_m).

In this instance, the results are good for a time $\sim 15T$ when a flow of mass outwards from the center makes the results untrustworthy. In this case, and in the isothermal case, the rise in $\langle \mu u \rangle$ and L^2 is also due to the fact that the μ step length is the largest of x, μ, u, and τ. However, it is clear that there is no fictious force (due to numerical errors) causing significant motion of the center of mass.

C. *Generalized Polytropes*

The distribution function is

$$\begin{aligned} f &= B(E_1 - \epsilon)^{n-3/2} A^{2m}, \qquad && \epsilon \leqslant E_1 \\ &= 0, && \epsilon \geqslant E_1, \end{aligned} \tag{33}$$

where ϵ is the particle energy (as before), B and E_1 are constants, and $A^2 = x^2 u^2(1 - \mu^2)$. Here,

$$\begin{aligned} g(x, u, \mu, 0) &= B'[w(x) - u^2]^{n-3/2}[x^2 u^2(1 - \mu^2)]^{2m}, \qquad && w(x) \geqslant u^2 \\ &= 0, && w(x) \leqslant u^2, \end{aligned} \tag{34}$$

where w satisfies [4],

$$(d/dx)(x^2(dw/dx)) = -x^{2m+2} w^{n+m}; \qquad w(0) = 1, w'(0) = 0. \tag{35}$$

This equation has the singular solution $w_s = a/x^p$ where

$$a^{n+m-1} = p(1 - p), \qquad p = 2(m + 1)/(n + m - 1). \tag{36}$$

This fact can be used to solve (35) analytically whenever $n = 3m + 5$. The result is

$$1/w = [1 + (x^{2(m+1)}/(2m + 3))]^{1/2(m+1)} \tag{37}$$

(compare with Plummer's model, i.e., set $m = 0$). One can also show (in general) that $i = m + 1$ for a generalized polytrope.

The last rows of Table I contain the result of a test with $m = 0.5$, $n = 6.5$. The crossing time is $T = 3.33 \times 10^{-2}$ (so $\Delta\tau = 0.3T$). The results are good for a time $\sim 15T$. The maximum density is at $x = 1$ (the direct opposite of the two previous tests) and there is a slow flow of mass towards the center. This model also has an m value near the region of unstable generalized polytropes [4].

D. *Other Tests*

A variety of other equilibrium configurations have been tested. In all cases, the results are very good for at least $5T$. In general, the less rapidly $g(x, u, \mu, 0)$ decreases with x, u, and μ the less accurate the results. Thus, if the initial condition is an $n = 1.5$ polytrope, the results are poorer than those obtained with an $n = 2.5$ polytrope.

IV. A Nonequilibrium Test

A common initial condition used in stellar dynamics calculations [8] is

$$g(x, u, \mu, 0) = \text{const.} \exp(-u^2 u_m{}^2/2), \tag{38}$$

where u_m and β are adjusted such that $K/\Omega = -1/4$ (i.e., one half of the equilibrium value). Thus, one expects the system to undergo a fairly rapid collapse. Previous work [5, 8] also indicates that the velocity distribution in the outer regions will become increasing anisotropic. The same coordinate grid used in Section III (with $u_m = 10$, $T = 6.65 \times 10^{-2}$, $\Delta\tau = 0.150T$) was used to evolve such a system.

The system does collapse and $|K/\Omega|$ rises. After $\sim 10T$, the central density (ρ_c) has increased by a factor of 100. The initially isotropic velocity distribution ($i = 1$) becomes increasingly anisotropic ($i = 0.7$ at $\tau = 0.65$), particularly in the outer regions. The quantity F,

$$F = \langle (\mu u)^4 \rangle - 3\langle (\mu u)^2 \rangle^2, \tag{39}$$

measures deviations from a Maxwellian velocity distribution [5]. Initially, $F = -1.04 \times 10^{-7}$, but it steadily increased to $F = 1.13$ at $\tau = 0.65$. Hence, there is an excess of high velocity particles relative to the Maxwell–Boltzmann distribution. The majority of these are also in the outer region (the "halo"). Finally, in Fig. 1, the density distribution at $\tau = 0.65$ is plotted. The full curve represents the isothermal distribution (IIIA) that best matches the numerical results at the origin.

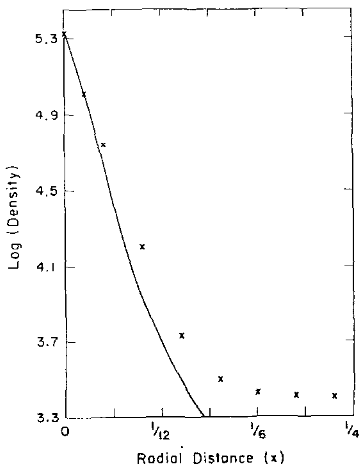

Fig. 1. The density distribution for the nonequilibrium model after ~ 10 crossing times. *The full curve shows the density distribution for an isothermal sphere.*

With the present numerical grid, the accuracy obtained here is poorer than for other reported work. However, the integration time does not depend on the number of particles in the system, nor have special assumptions regarding the physical state been made. Moreover, our knowledge of the evolution of self-gravitating systems is severely limited to a very few accurate integrations based on an even smaller set of initial conditions. With even the coarse grid used here, large scale experimentation with the initial conditions is possible.

Appendix

Fishman [3] first calculated the weights, w_k', and the values u_k for the Gaussian quadrature of

$$\int_0^1 u^2 f(u)\, du = \sum_{k=1}^{K} w_k' f(u_k). \tag{A1}$$

The largest value of K he considered was 8. The values for $\{u_k\}$, $\{w_k'\}$ when $K = 16$ are in Table AI.

TABLE AI

Numbers for Gaussian Quadrature[a]

k	u_k	w_k'
1	2.1393563 − 2	1.2778411 − 5
2	5.6775714 − 2	1.3757810 − 4
3	1.0631403 − 1	6.3446000 − 4
4	1.6844412 − 1	1.9235319 − 3
5	2.4117474 − 1	4.4948489 − 3
6	3.2217008 − 1	8.7475455 − 3
7	4.0882761 − 1	1.4803777 − 2
8	4.9836244 − 1	2.2357186 − 2
9	5.8789703 − 1	3.0612038 − 2
10	6.7455380 − 1	3.8348046 − 2
11	7.5554754 − 1	4.4112663 − 2
12	8.2827529 − 1	4.6506221 − 2
13	8.9039932 − 1	4.4496613 − 2
14	9.3992363 − 1	3.7685767 − 2
15	9.7525698 − 1	2.6460904 − 2
16	9.9527233 − 1	1.1999366 − 2

[a] The notation 1.23 − 4 means 1.23×10^{-4}.

The values for $\{\mu_j\}$, $\{w_j\}$ to be used in the evaluation of

$$\int_{-1}^{+1} f(\mu)\, d\mu = \sum_{j=1}^{J} w_j f(\mu_j) \tag{A2}$$

are the standard ones obtained by using Legendre polynomials.

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